SELF-SIMILAR SOLUTIONS OF PROBLEMS OF THE THEORY OF FILTRATION AND HEAT TRANSFER IN NONHOMOGENEOUS MEDIA

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The conditions for the existence of particular self-similar solutions of the equation of one-dimensional filtration in nonhomogeneous media are examined.

It is shown that the obtained self-similar solutions can be used in the investigation of direct and inverse problems of one-dimensional filtration and heat transfer.

1. The very different problems of one-dimensional unsteady filtration of a fluid in an inhomogeneous porous medium and of heat propagation are described by parabolic differential equations with corresponding boundary conditions and initial conditions. The equations have the form

$$\gamma^2(x)\frac{\partial U}{\partial t}=\frac{\partial}{x^\alpha}\frac{\partial}{\partial x}\left[x^\alpha\lambda^2(x)\frac{\partial U}{\partial x}\right] \qquad (\alpha=\text{const}) \tag{1.1}$$

Here x is a generalized space coordinate. The function $\gamma(x)$ is continuous and $\lambda(x)$ has also a continuous first derivative; $\alpha=0, 1, 2$ for plane-parallel, plane-radial and spherically symmetrical flows, respectively.

Let us represent the porosity m, the penetrability k, the density of the fluid σ and its kinematic viscosity ν in the form of following functions:

$$m=m_0(x)\,\Theta_m(p-p_0), \qquad k=k_0(x)\,\Theta_k(p-p_0) \tag{1.2}$$

$$\sigma=\sigma_0\Theta_\sigma(p-p_0), \qquad \nu=\nu_0\Theta_\nu(p-p_0)$$

Here p_0 is some fixed value of fluid pressure; the functions $m_0(x)$ and $k_0(x)$ determine the distribution of porosity and penetrability in the filtration region for $p=p_0$; σ_0 and ν_0 are values of density and viscosity of the fluid at $p=p_0$, the functions $\Theta_m(p-p_0)$, $\Theta_k(p-p_0)$, $\Theta_\sigma(p-p_0)$ and $\Theta_\nu(p-p_0)$ determine the dependence of parameters m, k, σ and ν on pressure, respectively.

We introduce the generalized function of Leibenzon Ψ, which is determined by the following equation:

$$\Psi=\int\frac{\Theta_k(p-p_0)}{\Theta_\nu(p-p_0)}\,d(p-p_0), \qquad \frac{k_0}{\nu_0}\frac{\partial\Psi}{\partial x}=-u \tag{1.3}$$

Here $u=\sigma v$ is the mass rate of filtration. From this and from the continuity equation, taking into accout (1.2), we have the filtration equation in the form

$$\sigma_0\nu_0 m_0(x)\frac{d(\Theta_m\Theta_\sigma)}{d\Psi}\frac{\partial\Psi}{\partial t}=\frac{1}{x^\alpha}\frac{\partial}{\partial x}\left[x^\alpha k_0(x)\frac{\partial\Psi}{\partial x}\right]$$

If the product $\Theta_m\Theta_\sigma$ can be assumed to be a linear function of the generalized Leibenzon function ψ, then the last equation is transformed into an equation of the form (1.1)

$$\gamma^2(x)\frac{\partial\Psi}{\partial t}=\frac{a}{x^\alpha}\frac{\partial}{\partial x}\left[x^\alpha\lambda^2(x)\frac{\partial\Psi}{\partial x}\right], \qquad \gamma^2(x)=\frac{m_0(x)}{\langle m_0\rangle} \tag{1.4}$$

$$\lambda^2(x)=\frac{k_0(x)}{\langle k_0\rangle}, \qquad a=\langle k_0\rangle(\sigma_0,\ \nu_0,\ \beta_n)^{-1}\langle m_0\rangle^{-1}, \qquad \beta_n=\frac{d}{d\Psi}(\Theta_m\Theta_\sigma)=\text{const}$$

Here $\langle m_0\rangle$ and $\langle k_0\rangle$ are averaged in the region of filtration values of penetrability and porosity for $p = p_0$.

Equations of the form (1.1) are encountered frequently in the investigation of thermal conductivity [1]. It is also possible to reduce to an equation of the form (1.1) problems of heat propagation in a homogeneous porous medium due to thermal conductivity and convective transfer taking into account the choking effect (Joule-Thomson effect) when water is forced into the layer.

If the rate of filtration is determined by the equation

$$U = U_0\, x^{-\alpha} \quad (U_0 = \text{const})$$

then the energy equation in a rigid filtration system [2] in the one-dimensional case can be reduced to the form

$$\frac{1}{x^\alpha}\frac{\partial}{\partial x}\left(x^\alpha k_T \frac{\partial T}{\partial x}\right) - \frac{C_f U_0}{x^\alpha}\frac{\partial T}{\partial x} = C_v \frac{\partial T}{\partial t} - \frac{A\nu U_0^2}{\sigma k x^{2\alpha}} \tag{1.5}$$

Here C_v is the volume heat capacity of the porous medium and the fluid which saturates it, C_f is the heat capacity of the fluid, k_T is the thermal conductivity of the porous medium and the fluid which saturates it, A = const is the thermal equivalent of work (thermal value).

It follows from the equation that the temperature T can be expressed in the form of a sum of functions $U(x, t)$ and some function $\Phi(x)$ which depends only on x. In this connection the function $U(x, t)$ is determined by Eq. (1.1) with $\gamma^2(x) = C_v$

$$\lambda^2(x) = k_T \exp \frac{-U_0 C_f}{k_T}\int x^{-\alpha}\, dx \qquad (\alpha = 1)$$

In the special case where $\gamma^2(x) \equiv 1$ and $\lambda^2(x) \equiv 1$, Eq. (1.1) corresponds to the equation of thermal conductivity in a homogeneous medium.

Let us examine the conditions which must be satisfied by functions $\gamma(x)$ and $\lambda(x)$ in order for Eq. (1.1) to have particular self-similar solutions of the form

$$U_\beta(x, t) = t^\beta y_\beta(\varepsilon), \quad \varepsilon = -\frac{\varphi(x)}{4at}, \quad \varphi(x) = \left(\int \frac{\gamma}{\lambda}\, dx + C_0\right)^2 \tag{1.6}$$

Here C_0 is a constant of integration, β is an arbitrary parameter.

Substitution of (1.6) into Eq. (1.1) and subsequent transformation, brings the latter to the following form:

$$\varepsilon \frac{d^2 y}{\partial \varepsilon^2} + \varepsilon \frac{dy}{d\varepsilon} + \frac{1}{2}\left[1 + \frac{d}{dx}\ln(x^\alpha \lambda \gamma) \Big/ \frac{d}{dx}\ln\left(\int \frac{\gamma}{\lambda}\, dx + C_0\right)\right]\frac{dy}{d\varepsilon} - \beta y = 0 \tag{1.7}$$

Solutions of Eq. (1.7) depend only on the variable ε and parameter β, if the expression in square brackets is equal to a constant (or zero).

Let us denote this constant through $(\alpha + b)$, where $b > 0$ is a positive parameter which characterizes the medium. Then after transformations we obtain two equations from (1.7)

$$x^\alpha \gamma \lambda = \left(\int \frac{\gamma}{\lambda}\, dx + C_0\right)^{\alpha + b - 1} \tag{1.8}$$

$$\varepsilon \frac{d^2 y}{d\varepsilon^2} + \left(\varepsilon + \frac{\alpha + b}{2}\right)\frac{dy}{d\varepsilon} - \beta y = 0 \tag{1.9}$$

After transformation we obtain from (1.8) the condition of self-similarity of (1.1) in the form

$$\left[(\alpha + b)\left(\int x^\alpha \gamma^2(x)\, dx + C_1\right)\right]^{2-\alpha-b} = \left[(2 - \alpha - b)\left(\int \frac{dx}{x^\alpha \lambda^2(x)} + C_2\right]^{\alpha + b}\right. \tag{1.10}$$

Here C_1 and C_2 are arbitrary constants. Equation (1.10) can easily be solved explicitly with respect to $\gamma(x)$ or $\lambda(x)$.

In connection with the continuity and positiveness of the quantity γ/λ, the function $\varphi(x)$ determined by the third equation (1.6) is bounded for bounded values of x. By varying the constant C_0 this function can be equated to zero only in one point of the interval of variation of x. This point will be designated by M. For the coordinate x to be expressed by the function $\varphi(x)$ uniquely, it is necessary to bring the point M in correspondence with the minimum value x_0 of the coordinate x in the given region.

The constant C_0 is determined by Eq. (1.8) for $\alpha + b \neq 1$ and it turns out to be arbitrary for $\alpha + b = 1$.

2. Let us formally determine two particular solutions of Eq. (1.1) by the following equalities:

$$y_\beta(\varepsilon) = \varepsilon^{1/2(2-\alpha-b)} e^{-\varepsilon} \varphi_2(\varepsilon) + \varphi_1(\varepsilon) \int_\varepsilon^\infty \eta^{-1/2(\alpha+b)} e^{-\eta}\, d\eta \tag{2.1}$$

$$y_s(\varepsilon) = e^{-\varepsilon} \psi_2(\varepsilon) + \varepsilon^{1/2(2-\alpha-b)} \psi_1(\varepsilon) \int_\varepsilon^\infty \eta^{1/2(\alpha+b-4)} e^{-\eta}\, d\eta$$

$$(2s = 2\beta - 2 + \alpha + b)$$

Functions y_β, y_s form a fundamental system if $\alpha + b \neq 2m$.

If we set in Eqs. (2.1) $\varphi_1(\varepsilon) \equiv 0$ or $\psi_1(\varepsilon) \equiv 0$, then $\varphi_2(\varepsilon)$ or $\psi_2(\varepsilon)$ are determined by the corresponding Pochhammer functions [3].

For $\beta = \pm n$ $(n = 0, 1, \ldots)$ functions $y_\beta(\varepsilon)$ will be generating functions for particular self-similar solutions $U_{\pm n}(\varphi, t)$ of the first kind for Eq. (1.1).

Correspondingly, for $s = \pm n$ the functions $y_s(\varepsilon)$ generate partial self-similar solutions $V_{\pm n}(\varphi, t)$ of the second kind for Eq. (1.1).

For $\beta = -n$ and $s = -n$ $(n = 1, 2, \ldots)$ we set $\varphi_1(\varepsilon) \equiv \psi_1(\varepsilon) \equiv 0$; the functions $\varphi_2(\varepsilon)$, $\psi_2(\varepsilon)$ are represented by the corresponding Pochhammer functions which will be polynomials in integral positive powers of ε.

In this connection the partial self-similar solutions of the first and second kind for Eq. (1.1) with negative indices $-n$ are determined by the equations

$$U_{-n}(\varphi, t) = t^{-n}\, {}^1/_2(2-\alpha-b)\left[1 + \sum_{i=1}^{n-1} \frac{(-2)^i (n-1)\ldots(n-i)\,\varepsilon^i}{i!\,(4-\alpha-b)\ldots(2i+2-\alpha-b)}\right] e^{-\varepsilon}$$

$$V_{-n}(\varphi, t) = t - {}^1/_2(2n-2+\alpha+b)\, e^{-\varepsilon}\left[1 + \sum_{i=1}^{n-1} \frac{(-2)^i (n-1)\ldots(n-i)\,\varepsilon^i}{i!\,(\alpha+b)\ldots(2i-2+\alpha+b)}\right] \tag{2.2}$$

Solutions U_{-n} and V_{-n} satisfy the conditions

$$U_{-n}(\varphi > 0, 0) = V_{-n}(\varphi > 0, 0) = 0, \qquad V_{-n}(0, t > 0) = t^{-1/2(2n-2+\alpha+b)}$$

$$U_{-n}(0, t > 0) = 0 \quad \text{for } \alpha + b < 2, \qquad U_{-n}(0, t > 0) = t^{-n} \quad \text{for } a + b = 2 \tag{2.3}$$

For $\beta = n$ and $s = n$ $(n = 0, 1, 2\ldots)$ the functions $\varphi_1(\varepsilon)$, $\varphi_2(\varepsilon)$, $\psi_1(\varepsilon)$ and $\psi_2(\varepsilon)$ are also represented as polynomials in integral positive powers of ε.

For $\beta = n$ the corresponding particular self-similar solutions $U_n(\varphi, t)$ of the first kind for Eq. (1.1) with positive indices n are determined by equations of the following form:

$$U_n(\varphi, t) = t^n \Big[\sum_{i=0}^{n} a_i \varepsilon^i \int_{\varepsilon}^{\infty} \eta^{-1/2(\alpha+b)} e^{-\eta} d\eta - e^{-\xi} \sum_{i=0}^{n-1} b_i \varepsilon^{1/2(2i+2-\alpha-b)} \tag{2.4}$$

$$a_0 = 1, \qquad a_i = \frac{2^i n (n-1) \ldots (n-i+1)}{i! (\alpha+b)(2+\alpha+b) \ldots (2i-2+\alpha+b)} \tag{2.5}$$

$$b_i = \frac{2(i+1) a_{i+1}}{n+i+1} + \frac{i+1}{n+i+1} \sum_{k=2}^{n-i} \frac{(i+2)\ldots(i+k)(2i+4-\alpha-b)\ldots(2i+2k-\alpha-b) a_{k+i}}{(n+i+2)\ldots(n+i+k)}$$

The solution of the first kind satisfies the conditions

$$U_n(\varphi > 0, 0) = 0 \tag{2.6}$$

$$U_n(0, t > 0) = t^n \Gamma[{}^1/_2(2-\alpha-b)] \qquad \text{for} \quad \alpha + b < 2$$

For $s = n$ the corresponding particular self-similar solutions $V_n(\varphi, t)$ of the second kind for Eq. **(1.1)** with positive indices n are determined by equations of the following form:

$$V_n(\varphi, t) = t^{\frac{2n+2-\alpha-b}{2}} \Big[\sum_{i=0}^{n} A_i \varepsilon^{\frac{2i+2-\alpha-b}{2}} \int_{\varepsilon}^{\infty} e^{-\eta} \eta^{\frac{\alpha+b-4}{2}} d\eta - e^{-\varepsilon} \sum_{i=0}^{n-1} B_i \varepsilon^i \Big] \tag{2.7}$$

$$A_0 = 1, \qquad A_i = \frac{2^i n (n-1) \ldots (n-i+1)}{i! (4-\alpha-b) \ldots (2i+2-\alpha-b)} \tag{2.8}$$

$$B_i = \frac{2(i+1) A_{i+1}}{n+i+1} + \frac{i+1}{n+i+1} \sum_{k=2}^{n-i} \frac{(i+2)\ldots(i+k)(2i+\alpha+b)\ldots(2i+2k-4+\alpha+b) A_{k+i}}{(n+i+2)\ldots(n+i+k)}$$

Functions $V_n(\varphi, t)$ satisfy the conditions

$$V_n(\varphi > 0, 0) = 0$$

$$V_n(0, t > 0) = t^{\frac{2n+2-\alpha-b}{2}} \left(\frac{2}{2-\alpha-b} - B_0 \right) \qquad \text{for} \quad \alpha + b < 2 \tag{2.9}$$

We introduce the functions $q(x, t)$ determined by the equations

$$q(x, t) = -x^{\alpha} \lambda^2(x) \frac{\partial U}{\partial x} \tag{2.10}$$

From this and from **(1.6)** after transformations we obtain

$$q(x, t) = -(4a)^{0.5(\alpha+b-2)} t^{\beta-1+0.5(\alpha+b)} \varepsilon^{0.5(\alpha+b)} dy/d\varepsilon \tag{2.11}$$

Functions $q(x, t)$, which are generated by substituting self-similar solutions of the first kind $U_{\pm n}(\varphi, t)$ into **(2.10)**, are designated through $q_{\pm n}(\varphi, t)$. When solutions of the second kind $V_{\pm n}(\varphi, t)$ are substituted, the functions are designated through $P_{\pm n}(\varphi, t)$.

The functions $q_n(\varphi, t)$ and $p_n(\varphi, t)$ $(n = 0, 1, \ldots)$ satisfy the following conditions, respectively:

$$p_n(0, t > 0) = 2t^n (4a)^{1/2(\alpha+b-2)} \Gamma\left(\frac{\alpha+b}{2}\right), \qquad p_n(\varphi > 0, 0) = q_n(\varphi > 0, 0) = 0 \tag{2.12}$$

$$q_n(\varphi, t) \equiv p_n(\varphi, t) \qquad \text{for} \quad \alpha + b = 2$$

3. It follows from **(2.6)** and **(2.12)** that $U_n(0, t > 0)$ for $\alpha + b < 2$, and $p_n(0, t > 0)$ for $\alpha + b > 0$ form complete systems of functions.

We shall seek solutions $U(x, t)$, bounded in the finite interval of time, for Eq. **(1.1)** on a semi-infinite straight line in space $(\alpha = 0)$, inside a circle of infinite radius

($\alpha = 1$) or inside a sphere of infinite radius ($\alpha = 2$) with the initial conditions

$$U(x, 0) = F(x) + \sum_{m=1} A_m{}^\circ U_0(\varepsilon_m), \qquad \varepsilon_m = \frac{\varphi(x)}{4aT_m} \qquad (T_m = \mathrm{const})$$
$$F(x) = C_5 + C_6 \int \frac{dx}{x^\alpha \lambda^2(x)} \tag{3.1}$$

Here C_5 and C_6 are arbitrary constants.

The boundary conditions are given by one of the equations

$$U(0, t) = F(0) + \theta_1(t), \quad x^\alpha \lambda^2(x) dU/dx|_{x=0} = C_4 - \theta_2(t) \tag{3.2}$$

Here $\theta_1(t)$ and $\theta_2(t)$ are continuous functions. We search for the solution in the form of a sum of particular solutions $U_n(\varphi, t)$, $V_n(\varphi, t)$ and a fundamental function which satisfies the initial condition (3.1)

$$U(x, t) = F(x) + \sum_{m=1} A_m{}^\circ U_0\left[\frac{\varphi(x)}{4a(T_m + t)}\right] + \sum_{n=0} B_n{}^\circ U_n(\varphi, t) + \sum_{n=0} D_n V_n(\varphi, t) \tag{3.3}$$

If the boundary conditions are given by the second equation (3.2) for $\alpha + b = 2$ or by the first equation (3.2), then we assume that all coefficients D_n in (3.3) are equal to zero. But if the conditions are given by the second equation (3.2) for $\alpha + b \neq 2$, then we assume that all coefficients $B_n{}^\circ$ and $A_m{}^\circ$ are equal to zero.

By virtue of (2.6), (3.9) and (2.12) the solutions (3.3) satisfy the initial condition (3.1). The boundary conditions (3.2) are reduced to the following form, respectively:

$$\Gamma[0.5(2 - \alpha - b)] \sum_{n=0} B_n{}^\circ t^n = \Theta_1(t) - \Gamma[0.5(2 - \alpha - b)] \sum_{m=1} A_m{}^\circ \qquad (\alpha + b < 2)$$

$$2 \sum_{n=0} B_n{}^\circ \left[\sum_{i=0}^{n} (a_i + ib_0)\right] t^n = \Theta_2(t) - 2 \sum_{m=1} A_m{}^\circ \qquad (\alpha + b = 2)$$

$$2(4a)^{1/2(\alpha+b-2)} \Gamma[0.5(\alpha + b)] \sum D_n t^n = \Theta_2(t) \qquad (\alpha + b \neq 2) \tag{3.4}$$

By virtue of the Weierstrass theorem it is possible in solution (3.3) to dispose of constants $B_n{}^\circ$ in the presence of $D_n{}^\circ$ and of constants D_n in the presence of $B_n{}^\circ = 0$ in such a manner that the conditions (3.4) are satisfied with prescribed accuracy.

Functions U_{-n} and V_{-n} can be used for asymptotic expansion of various solutions of Eq. (1.1) for $t \to \infty$. In the interval $[x_0, \infty)$, $(0 \leqslant x_0)$ these solutions satisfy initial conditions (2.8) if the value $x = x_0$ is such that starting with some $t = t_0$, the function $\varphi(x_0) = \varphi_0$ in the third equation (1.6) satisfies the inequality

$$\varphi_0 \ll 4at, \qquad t > t_0 \tag{3.5}$$

When (3.5) is satisfied, the functions $U_n(\varphi_0, t)$ are determined in accordance with (2.4) and (2.5) by the following approximate equations:

$$t^{-n} U_n(\varphi_0, t > t_0) \approx \Gamma\left(1 - \frac{\alpha + b}{2}\right) - \left(\frac{2}{2 - \alpha - b} + b_0\right)\left(\frac{\varphi_0}{4at}\right)^{1/2(2-\alpha-b)} \qquad (\alpha + b < 2)$$

$$t^{-n} U_n(\varphi_0, t > t_0) \approx \ln t - \ln \frac{\varphi_0}{2.25a} - b_0 \qquad (\alpha + b = 2)$$

$$t^{-n} U_n(\varphi_0, t > t_0) \approx \left(\frac{2}{a + b - 2} - b_0\right)\left(\frac{\varphi_0}{4at}\right)^{1/2(2-\alpha-b)}$$
$$- \frac{2}{\alpha + b - 2} \Gamma\left(2 - \frac{\alpha + b}{2}\right) \qquad (\alpha + b > 2) \tag{3.6}$$

When (3.5) is satisfied, the functions U_{-n} and V_{-n} are determined with sufficient accuracy by equations

$$U_{-n}(\varphi_0, t) \approx \left(\frac{\varphi_0}{4a}\right)^{1/2(2-\alpha-b)} t^{-1/2(2n+2-\alpha-b)}, \qquad V_{-n}(\varphi_0, t) \approx t^{-1/2(2n-2+\alpha+b)} \tag{3.7}$$

Let the solution of Eq. (1.1), which is being investigated and which satisfies the initial conditions (3.1), be formally determined by the equation

$$U(x, t) = F(x) + \sum_{m=0} A_m{}^\circ U_0\left[\frac{\varphi(x)}{4a(T_m+t)}\right] + U^*(x, t)\, t^{(\alpha+b-2)\,1/2} \tag{3.8}$$

Here the function $U^*(x, t)$, which is a regular solution of Eq. (1.1), satisfies the conditions

$$U^*(x, 0) = 0, \qquad U^*(x_0, t)\, t^{-1/2\,|2-\alpha-b|} \to 0 \quad \text{for } t \to \infty \tag{3.9}$$

In accordance with theorems on expansion of regular functions into asymptotic series [4], the function $U^*(x_0, t)$ is expanded for $t \to \infty$ into an asymptotic power series which converges to $U^*(x_0, t)$ for sufficiently large t

$$U(x_0, t) \sim \frac{c_1}{t} + \frac{c_2}{t^2} + \ldots \qquad (c_n = \text{const})$$

Therefore, by virtue of conditions (3.7) and (3.8), the function $U^*(x, t)$ for $t > t_0$ can be expressed in the form

$$\begin{aligned} &\left(\frac{\varphi}{4at}\right)^{0.5(2-\alpha-b)} U^*(x, t) \approx \sum_{n=1} c_n U_{-n}(\varphi, t) \qquad (\alpha+b<2) \\ &t^{1/2(2-\alpha-b)}\, U^*(x, t) \approx \sum_{n=1} c_n V_{-n}(\varphi, t) \qquad (\alpha+b \geqslant 2) \end{aligned} \tag{3.10}$$

In paper [5] an effective method for evaluating parameters of homogeneous porous layers (strata) from pressure recovery curves in the case of non-instantaneous stoppage of fluid supply to the pore (fissure) is based on asymptotic expansion.

Utilizing self-similar solutions of the second kind, it is easy to generalize this method also to inhomogeneous layers.

It follows from (2.7) and (2.12) that the face pressure recovery curve in the infinite layer tends asymptotically to the semi-logarithmic straight line only for $\alpha + b = 2$.

For $\alpha + b < 2$, the face pressure in the stopped pore in an infinite layer is a power function of time.

For $\alpha + b > 2$, the face pressure in the stopped pore in an infinite layer approaches some constant.

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